

Dirichlet problem associated with Dunkl Laplacian on W -invariant open sets

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Abstract

Combining probabilistic and analytic tools from potential theory, we investigate Dirichlet problems associated with the Dunkl Laplacian Δ_k . We establish, under some conditions on the open set $D \subset \mathbb{R}^d$, the existence of a unique continuous function h in the closure of D , twice differentiable in D , such that

$$\Delta_k h = 0 \quad \text{in } D \quad \text{and} \quad h = f \quad \text{on } \partial D.$$

We also give a probabilistic formula characterizing the solution h . The function f is assumed to be continuous on the Euclidean boundary ∂D of D .

1 Introduction

In their monograph [2], J. Bliedtner and W. Hansen developed four descriptions of potential theory using balayage spaces, families of harmonic kernels, sub-Markov semigroups and Markov processes. They proved that all these descriptions are equivalent and gave a straight presentation of balayage theory which is, in particular, applied to the generalized Dirichlet problem associated with a large class of differential and pseudo-differential operators.

Let W be a finite reflection group on \mathbb{R}^d , $d \geq 1$, with root system R and we fix a positive subsystem R_+ of R and a nonnegative multiplicity function $k : R \rightarrow \mathbb{R}_+$. For every $\alpha \in R$, let H_α be the hyperplane orthogonal to α and σ_α be the reflection with respect to H_α , that is, for every $x \in \mathbb{R}^d$,

$$\sigma_\alpha x = x - 2 \frac{\langle x, \alpha \rangle}{|\alpha|^2} \alpha$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product of \mathbb{R}^d . C. F. Dunkl introduced in [4] the operator

$$\Delta_k = \sum_{i=1}^d T_i^2,$$

which will be called later *Dunkl Laplacian*, where, for $1 \leq i \leq d$, T_i is the differential-difference operator defined for $f \in C^1(\mathbb{R}^d)$ by

$$T_i f(x) = \frac{\partial f}{\partial x_i}(x) + \sum_{\alpha \in R_+} k(\alpha) \alpha_i \frac{f(x) - f(\sigma_\alpha x)}{\langle \alpha, x \rangle}.$$

Our main goal in this paper is to investigate the *Dirichlet problem* associated with the Dunkl Laplacian. More precisely, given a bounded open set $D \subset \mathbb{R}^d$ and a continuous real-valued function f on $D^c := \mathbb{R}^d \setminus D$, we are concerned with the following problem:

$$\begin{cases} \Delta_k h = 0 & \text{in } D, \\ h = f & \text{on } D^c. \end{cases} \quad (1)$$

We mean by a solution of (1) every function $h : \mathbb{R}^d \rightarrow \mathbb{R}$ which is continuous in \mathbb{R}^d , twice differentiable in D and such that both equations in (1) are pointwise fulfilled. In the particular case where D is the unit ball of \mathbb{R}^d , M. Maslouhi and E. H. Youssfi [11] solved problem (1) by methods from harmonic analysis using the Poisson kernel for Δ_k which is introduced by C. F. Dunkl and Y. Xu [5]. It should be noted that, for balls with center $a \neq 0$, the Poisson kernel for Δ_k is not known up to now.

Let us briefly introduce our approach. It is well known (see [6] and references therein) that there exists a càdlàg \mathbb{R}^d -valued Markov process

$$X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P^x),$$

which is called *Dunkl process*, with infinitesimal generator $\frac{1}{2}\Delta_k$. For a given bounded Borel function $h : \mathbb{R}^d \rightarrow \mathbb{R}$, we define

$$H_U h(x) = E^x[h(X_{\tau_U})]$$

for every $x \in \mathbb{R}^d$ and every bounded open subset U of \mathbb{R}^d , where

$$\tau_U = \inf\{t > 0; X_t \notin U\}$$

denotes the first exit time from U by X . We first show that if h is continuous in \mathbb{R}^d and twice differentiable in D then $\Delta_k h = 0$ in D if and only if h is X -harmonic in D , i.e., $H_U h(x) = h(x)$ for every open set U such that $\overline{U} \subset D$ (we shall write $U \Subset D$) and for every $x \in U$. We then conclude, using the general framework of balayage spaces [2], that problem (1) admits at most one solution. Moreover, if the open set D is regular for the Dunkl process, then $H_D f$ will be the solution of (1) provided it is of class C^2 in D .

For some examples of Markov processes, namely Brownian motion or α -stable process, some additional geometric assumptions on the Euclidean boundary ∂D of D permit a decision on the regularity of D . In fact, it is well known that D is regular, with respect to Brownian motion or α -stable process, whenever each boundary point of D satisfies the "cone condition". For a particular choice of the root system R , we shall prove in Section 3 that the cone condition is still sufficient for the regularity of D with respect to the Dunkl process. However, we could not know whether this result holds true for arbitrary root systems. In this setting, we only show that balls of center 0 are regular.

Finally, assuming that D is regular, the study of problem (1) is equivalent to the study of smoothness of $H_D f$. Indeed, as was mentioned above, (1) has a solution if and only if

$$H_D f \in C^2(D).$$

To that end, we need to assume that D is W -invariant which means that $\sigma_\alpha(D) \subset D$ for every $\alpha \in R$. Hence, using the fact that the operator Δ_k is hypoelliptic in D (see [7, 10]) we prove that $H_D f$ is infinitely differentiable in D . Thus, we not only deduce the existence and uniqueness of the solution to

$$\begin{cases} \Delta_k h &= 0 & \text{in } D, \\ h &= f & \text{on } \partial D, \end{cases} \quad (2)$$

but we also prove that h is given by the formula $h(x) = E^x[f(X_{\tau_D})]$.

Throughout this paper, let $\lambda = \gamma + \frac{d}{2} - 1$ and assume that $\lambda > 0$.

2 Harmonic Kernels

For the sake of simplicity, we assume in all the following that $|\alpha|^2 = 2$ for every $\alpha \in R$. It follows from [4] that, for $f \in C^2(\mathbb{R}^d)$,

$$\Delta_k f(x) = \Delta f(x) + 2 \sum_{\alpha \in R_+} k(\alpha) \left(\frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \right), \quad (3)$$

where Δ denotes the usual Laplacian on \mathbb{R}^d . M. Rösler has shown in [13] that $\frac{1}{2}\Delta_k$ generates a Feller semigroup $P_t^k(x, dy) = p_t^k(x, y)w_k(y)dy$ which has the expression

$$p_t^k(x, y) = \frac{1}{c_k t^{\gamma + \frac{d}{2}}} \exp\left(-\frac{|x|^2 + |y|^2}{2t}\right) E_k\left(\frac{x}{\sqrt{t}}, \frac{y}{\sqrt{t}}\right), \quad (4)$$

where $E_k(\cdot, \cdot)$ is the Dunkl kernel associated with W and k (see [5]), the constant c_k is taken such that $P_1^k 1 \equiv 1$, $\gamma = \sum_{\alpha \in R_+} k(\alpha)$ and w_k is the W -invariant weight function defined on \mathbb{R}^d by

$$w_k(y) = \prod_{\alpha \in R_+} |\langle y, \alpha \rangle|^{2k(\alpha)}.$$

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, X_t, P^x)$ be the Dunkl process in \mathbb{R}^d with transition kernel $P_t^k(x, dy)$. For every bounded open subset D of \mathbb{R}^d , let τ_D be the first exit time from D by X . A point $z \in \partial D$ is said to be *regular* (for D) if $P^z[\tau_D = 0] = 1$ and *irregular* if $P^z[\tau_D = 0] = 0$. Notice that by Blumenthal's zero-one law, each boundary point of D is either regular or irregular. It is also easy verified that the fact that Dunkl process has right continuous paths yields that $P^x[\tau_D = 0] = 0$ if $x \in D$ and $P^x[\tau_D = 0] = 1$ if $x \in \mathbb{R}^d \setminus \overline{D}$.

Proposition 1. $E^x[\tau_D] < \infty$ for every $x \in \mathbb{R}^d$ and every bounded open subset D of \mathbb{R}^d .

Proof. Let D be a bounded open subset of \mathbb{R}^d , $x \in \mathbb{R}^d$ and choose $r > 0$ such that the ball $B = B(0, r)$ contains x and D . Then, applying Fubini's theorem and using spherical coordinates,

$$\begin{aligned} E^x[\tau_B] &\leq \int_0^\infty E^x[\mathbf{1}_B(X_s)] ds \\ &= \int_0^r t^{2\lambda+1} \int_0^\infty \int_{S^{d-1}} p_s^k(x, tz) w_k(z) \sigma(dz) ds dt. \end{aligned}$$

Here and in all the following, σ denotes the surface area measure on the unit sphere S^{d-1} of \mathbb{R}^d . It is well known (see [13, 14]) that for every $x, y \in \mathbb{R}^d$ and $s > 0$,

$$p_s^k(x, y) = \frac{1}{c_k^2} \int_{\mathbb{R}^d} e^{-\frac{s}{2}|\xi|^2} E_k(-ix, \xi) E_k(iy, \xi) w_k(\xi) d\xi$$

and

$$\int_{S^{d-1}} E_k(ix, \xi) w_k(\xi) \sigma(d\xi) = \frac{c_k}{2\lambda\Gamma(\lambda+1)} j_\lambda(|x|),$$

where

$$j_\lambda(z) := \Gamma(\lambda+1) \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{4^n n! \Gamma(n+\lambda+1)}$$

is the Bessel normalized function. Hence

$$\begin{aligned} E^x[\tau_D] &\leq \int_0^\infty E^x[\mathbf{1}_B(X_s)] ds \\ &= \frac{1}{2^{2\lambda-1}(\Gamma(\lambda+1))^2} \int_0^r t^{2\lambda+1} \int_0^\infty j_\lambda(ut) j_\lambda(u|x|) u^{2\lambda-1} du dt \\ &= \frac{2^{2\lambda-1} \Gamma(\lambda+1) \Gamma(\lambda)}{2^{2\lambda-1} (\Gamma(\lambda+1))^2} \int_0^r t^{2\lambda+1} (\max(t, |x|))^{-2\lambda} dt \end{aligned} \quad (5)$$

$$= \frac{r^2}{2\lambda} - \frac{|x|^2}{2\lambda+2} < \infty. \quad (6)$$

In order to get (5) one should think about formula (11.4.33) in [1]. \square

Let D be a bounded open subset of \mathbb{R}^d . For every $x \in \mathbb{R}^d$, the exit distribution $H_D(x, \cdot)$ from D by the Dunkl process starting at x will be called *harmonic measure* relative to x and D . That is, for every Borel subset A of \mathbb{R}^d ,

$$H_D(x, A) = P^x(X_{\tau_D} \in A).$$

It is clear that $H_D(x, \cdot) = \delta_x$ the Dirac measure at x whenever $x \in \partial D$ is regular or $x \notin \overline{D}$. We define

$${}^W D := \cup_{w \in W} w(D) \quad \text{and} \quad \Gamma_D := \overline{{}^W D} \setminus D.$$

In other words, ${}^W D$ is the smallest open set containing D which is invariant under the reflection group W . The following theorem ensures that $H_D(x, \cdot)$ is supported by Γ_D for every $x \in \overline{D}$.

Theorem 2. *Let D be a bounded open subset of \mathbb{R}^d . Then for every $x \in \overline{D}$,*

$$P^x(X_{\tau_D} \in \Gamma_D) = 1. \quad (7)$$

Proof. It is easily seen that for every regular boundary point x , $P^x(X_{\tau_D} \in \Gamma_D) = \delta_x(\Gamma_D) = 1$. Now, assume that $x \in D$ or $x \in \partial D$ is irregular and consider the function F defined for every $y, z \in \mathbb{R}^d$ by $F(y, z) = 0$ if $z \in \{\sigma_\alpha y; \alpha \in R_+\}$ and $F(y, z) = 1$ otherwise. Let

$$Y_t := \sum_{s < t} \mathbf{1}_{\{X_{s-} \neq X_s\}} F(X_{s-}, X_s), \quad t > 0.$$

It follows from [6, Proposition 3.2] that for every $t > 0$, $P^x(Y_t = 0) = 1$ and consequently

$$P^x(\mathbf{1}_{\{X_{s-} \neq X_s\}} F(X_{s-}, X_s) = 0; \forall s > 0) = 1.$$

Then, since $P^x(0 < \tau_D < \infty) = 1$ we deduce that

$$P^x \left(\mathbf{1}_{\{X_{\tau_D^-} \neq X_{\tau_D}\}} F(X_{\tau_D^-}, X_{\tau_D}) = 0 \right) = 1.$$

On the other hand, seeing that $X_{\tau_D^-} \in \overline{D}$ on $\{0 < \tau_D < \infty\}$ we have

$$\{X_{\tau_D} \notin \Gamma_D, 0 < \tau_D < \infty\} \subset \left\{ \mathbf{1}_{\{X_{\tau_D^-} \neq X_{\tau_D}\}} F(X_{\tau_D^-}, X_{\tau_D}) = 1 \right\}.$$

This finishes the proof. \square

Let \mathcal{O} be the set of all bounded open subsets of \mathbb{R}^d . In the following, we denote by $\mathcal{B}_b(\mathbb{R}^d)$ the set of all bounded Borel measurable functions on \mathbb{R}^d . For every $D \in \mathcal{O}$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$, let $H_D f$ be the function defined on \mathbb{R}^d by

$$H_D f(x) = E^x [f(X_{\tau_D})] = \int f(y) H_D(x, dy).$$

Since X is a Hunt process, it follows from the general framework of balayage spaces studied by J. Bliedtner and W. Hansen in [2] that, for every $D \in \mathcal{O}$ and $f \in \mathcal{B}_b(\mathbb{R}^d)$ with compact support, $H_D f$ is continuous in D and for every $V \Subset D$,

$$H_V H_D = H_D \quad \text{in } V. \quad (8)$$

Since $\text{supp } H_D(x, \cdot) \subset \Gamma_D$ for every $x \in \overline{D}$, it is trivial that

$$H_D f(x) = H_D(1_{\Gamma_D} f)(x), \quad x \in \overline{D}.$$

Hence, we immediately conclude that $H_D f$ is continuous in D . For every $D \in \mathcal{O}$ and every $f \in \mathcal{B}_b(\Gamma_D)$, it will be convenient to denote again

$$H_D f(x) = \int f(y) H_D(x, dy), \quad x \in \overline{D}. \quad (9)$$

Let U be an open subset of \mathbb{R}^d . A locally bounded function $h : {}^W U \rightarrow \mathbb{R}$ is said to be X -harmonic in U if $H_D h(x) = h(x)$ for every open set $D \Subset U$ and every $x \in D$. If U is bounded and h is continuous in \overline{WU} then h is X -harmonic in U if and only if for every $x \in U$,

$$h(x) = H_U h(x). \quad (10)$$

In fact, let $x \in U$ and let $(U_n)_{n \geq 1}$ be a sequence of nonempty bounded open subsets of \mathbb{R}^d such that $x \in U_n \Subset U_{n+1}$ and $U = \cup_n U_n$. Then $(\tau_{U_n})_n$ converges to τ_U almost surely. Hence, the continuity of h on \overline{WU} together with the quasi-left-continuity of the Dunkl process yield that $H_U h(x) = \lim_n H_{U_n} h(x)$. The following proposition follows immediately from (10).

Proposition 3. *Let $U \in \mathcal{O}$ and let h be a continuous function on \overline{WU} . If h is X -harmonic in U , then*

$$\max_{x \in \overline{WU}} h(x) = \max_{x \in \Gamma_U} h(x) \quad \text{and} \quad \min_{x \in \overline{WU}} h(x) = \min_{x \in \Gamma_U} h(x).$$

We shall denote by G^k the Green function of Δ_k which is defined for every $x, y \in \mathbb{R}^d$ by

$$G^k(x, y) = \int_0^\infty p_t^k(x, y) dt.$$

Since p_t^k is symmetric in $\mathbb{R}^d \times \mathbb{R}^d$, we obviously see that the Green function G^k is also symmetric in $\mathbb{R}^d \times \mathbb{R}^d$. Therefore, it follows from [3, Theorem VI.1.16] that for every $D \in \mathcal{O}$ and for every $x, y \in \mathbb{R}^d$,

$$\int G^k(x, z) H_D(y, dz) = \int G^k(y, z) H_D(x, dz). \quad (11)$$

Furthermore, for every $y \in \mathbb{R}^d$, the function $G^k(\cdot, y)$ is excessive, that is, $G^k(\cdot, y)$ is lower semi-continuous in \mathbb{R}^d and $\int p_t^k(x, z) G^k(z, y) w_k(z) dz \leq G^k(x, y)$ for every $t > 0$ and $x \in \mathbb{R}^d$. Consequently, it follows from [2, Theorem IV.8.1] that $G^k(\cdot, y)$ is hyperharmonic on \mathbb{R}^d , i.e., for every $D \in \mathcal{O}$ and for every $x \in \mathbb{R}^d$,

$$\int G^k(z, y) H_D(x, dz) \leq G^k(x, y). \quad (12)$$

Lemma 4. *Let $f \in C_c^2(\mathbb{R}^d)$ and $D \in \mathcal{O}$. For every $x \in \mathbb{R}^d$,*

$$\int G^k(x, y) \Delta_k f(y) w_k(y) dy = -2f(x). \quad (13)$$

In particular,

$$H_D f(x) - f(x) = \frac{1}{2} E^x \left[\int_0^{\tau_D} \Delta_k f(X_s) ds \right]. \quad (14)$$

Proof. To get (13) it suffices to recall that

$$\frac{\partial}{\partial t} P_t^k = \frac{1}{2} P_t^k \Delta_k, \quad t > 0.$$

Then, we integrate over t and use the fact that $\lim_{t \rightarrow 0} P_t^k f(x) = f(x)$ and $\lim_{t \rightarrow \infty} P_t^k f(x) = 0$ for every $x \in \mathbb{R}^d$. Formula (14) follows from (13) and the strong Markov property. \square

Let U be an open subset of \mathbb{R}^d . A function $h : {}^W U \rightarrow \mathbb{R}$ is said to be Δ_k -harmonic in U if $h \in C^2(U)$ and $\Delta_k h(x) = 0$ for every $x \in U$.

Theorem 5. *Let U be an open subset of \mathbb{R}^d and let $h \in C({}^W U)$. If $h \in C^2(U)$ then h is Δ_k -harmonic in U if and only if h is X -harmonic in U .*

Proof. Let $D \Subset U$ and let $x \in D$. Then

$$H_D h(x) - h(x) = \frac{1}{2} E^x \left[\int_0^{\tau_D} \Delta_k h(X_s) ds \right]. \quad (15)$$

In fact, choose an open set V such that $D \Subset V \Subset U$, $f \in C_c^2(\mathbb{R}^d)$ which coincides with h in V and let $\psi = h - f$. Then using (14) we obtain

$$H_D h(x) - h(x) = \frac{1}{2} E^x \left[\int_0^{\tau_D} \Delta_k f(X_s) ds \right] + H_D \psi(x). \quad (16)$$

For every $y \in \mathbb{R}^d$, let $N(y, dz)$ be the Lévy kernel of the Dunkl process X which is given by the following formula [6]

$$N(y, dz) = \sum_{\alpha \in R_+, \langle y, \alpha \rangle \neq 0} \frac{k(\alpha)}{\langle \alpha, y \rangle^2} \delta_{\sigma_\alpha y}(dz). \quad (17)$$

Since $\psi = 0$ on V , it follows from [8, Theorem 1] that

$$H_D \psi(x) = E^x \left[\int_0^{\tau_D} \int \psi(z) N(X_s, dz) ds \right]. \quad (18)$$

On the other hand, by (3) and (17) we easily see that for every $y \in D$,

$$\Delta_k f(y) = \Delta_k h(y) - 2 \int \psi(z) N(y, dz). \quad (19)$$

Thus formula (15) is obtained by combining (16), (18) and (19) above. Now, h is obviously X -harmonic in U whenever it is Δ_k -harmonic in U . Conversely, assume that h is X -harmonic in U and let $x \in U$. Since $h \in C(WU) \cap C^2(U)$ then $\Delta_k h$ is continuous in U and consequently for every $\varepsilon > 0$ there exists an open neighborhood $D \Subset U$ of x such that $|\Delta_k h(y) - \Delta_k h(x)| \leq \varepsilon$ for every $y \in D$. Using formula (15), we obtain

$$|\Delta_k h(x)| = \frac{1}{E^x[\tau_D]} \left| E^x \left[\int_0^{\tau_D} (\Delta_k h(X_s) - \Delta_k h(x)) ds \right] \right| \leq \varepsilon.$$

Hence $\Delta_k h(x) = 0$ as desired. □

3 Regular Sets

A bounded open subset D of \mathbb{R}^d is said to be *regular* if each $z \in \partial D$ is regular for D . A complete study of regularity is developed by J. Bliedtner and W. Hansen in [2]. It follows that a point $z \in \partial D$ is regular for D if and only if for every $f \in C(\Gamma_D)$,

$$\lim_{x \in D, x \rightarrow z} H_D f(x) = f(z).$$

Consequently, $H_D f$ is continuous on \overline{D}^W whenever D is regular and $f \in C(\Gamma_D)$.

Example 6. For all $R > r > 0$, the ball $B(0, R)$ and the annulus $C(r, R) = \{x \in \mathbb{R}^d; r < \|x\| < R\}$ are regular.

In fact, by [2, Proposition VII.3.3], it is sufficient to find a neighborhood V of $z \in \partial D$ and a real function u such that

- i) u is positive in $V \cap D$,
- ii) u is X -harmonic in $V \cap D$,
- iii) $\lim_{x \in V \cap D, x \rightarrow z} u(x) = 0$.

Consider $V = \mathbb{R}^d \setminus \{0\}$ and g the function defined on V by

$$g(x) = \frac{1}{|x|^{2\lambda}}.$$

Using formula (3), simple computation shows that g is Δ_k -harmonic in V which yields, by theorem 5, that g is X -harmonic in V . Let $z \in \mathbb{R}^d$ such that $|z| = R$ and consider

$$u(x) = g(x) - \frac{1}{R^{2\lambda}}, \quad x \in V.$$

It is clear that u satisfy (i), (ii) and (iii) above with $D = B(0, R)$ or $D = C(r, R)$. Hence z is regular for D . Similarly, taking

$$u(x) = \frac{1}{r^{2\lambda}} - g(x), \quad x \in V,$$

we conclude that all points $z \in \mathbb{R}^d$ such that $|z| = r$ are regular for $C(r, R)$.

A sufficient condition for regularity, known as the cone condition, is given in the following theorem for a particular root system R .

Theorem 7. *Let (e_1, \dots, e_d) be the canonical basis of \mathbb{R}^d and consider the root system $R = \{\pm e_i, 1 \leq i \leq d\}$. Let D be a bounded open subset of \mathbb{R}^d and let $z \in \partial D$. Assume that there exists a cone C of vertex z such that $C \cap B(z, r) \subset D^c$ for some $r > 0$. Then z is regular for D .*

Proof. It is trivial that $P^z[\tau_D \leq t] \geq P^z[X_t \in C \cap B(z, r)]$ for all $t > 0$. Therefore, in virtue of Blumenthal's zero-one law, it is sufficient to show that $\liminf_{t \rightarrow 0} P^z[X_t \in C \cap B(z, r)]$ is positive. Denote $C_0 = C - z$, then

$$\begin{aligned} P^z[X_t \in C \cap B(z, r)] &= \int_{C \cap B(z, r)} p_t^k(z, y) w_k(y) dy \\ &= \frac{1}{t^\gamma} \int_{C_0 \cap B(0, \frac{r}{\sqrt{t}})} p_1^k\left(\frac{z}{\sqrt{t}}, \frac{z}{\sqrt{t}} - y\right) w_k(z - \sqrt{t}y) dy. \end{aligned} \quad (20)$$

It is trivial to see, from (4), that

$$p_1^k\left(\frac{z}{\sqrt{t}}, \frac{z}{\sqrt{t}} - y\right) = e^{-\frac{|y|^2}{2}} e^{-\langle \frac{z}{t}, z - \sqrt{t}y \rangle} E_k\left(\frac{z}{t}, z - \sqrt{t}y\right).$$

Let $k_i = k(e_i)$ and $y_i = \langle y, e_i \rangle$ for every $y \in \mathbb{R}^d$ and $i \in \{1, \dots, d\}$. It is known [16] that for all $x, y \in \mathbb{R}^d$,

$$e^{-\langle x, y \rangle} E_k(x, y) = \prod_{i=1}^d M(k_i, 2k_i + 1, -2x_i y_i).$$

$M(k_i, 2k_i + 1, \cdot)$ denotes the Kummer's function defined on \mathbb{R} by

$$M(k_i, 2k_i + 1, s) = \sum_{n \geq 0} \frac{(k_i)_n}{(2k_i + 1)_n} \frac{s^n}{n!} = 1 + \frac{k_i}{2k_i + 1} s + \frac{k_i(k_i + 1)}{(2k_i + 1)(2k_i + 2)} \frac{s^2}{2!} + \dots$$

Therefore, for any $y \in \mathbb{R}^d$ and $t > 0$, we have

$$\begin{aligned} & \frac{1}{t^\gamma} e^{-\langle \frac{z}{t}, z - \sqrt{t}y \rangle} E_k \left(\frac{z}{t}, z - \sqrt{t}y \right) w_k \left(z - \sqrt{t}y \right) \\ &= \prod_{i=1}^d \frac{M(k_i, 2k_i + 1, -2\frac{z_i}{t}(z_i - \sqrt{t}y_i)) (z_i - \sqrt{t}y_i)^{2k_i}}{t^{k_i}}. \end{aligned}$$

First, it is clear that

$$\frac{1}{t^{k_i}} M(k_i, 2k_i + 1, -2\frac{z_i}{t}(z_i - \sqrt{t}y_i)) (z_i - \sqrt{t}y_i)^{2k_i} = \begin{cases} 1 & \text{if } k_i = 0 \\ y_i^{2k_i} & \text{if } z_i = 0. \end{cases}$$

Next, assume that $k_i > 0$ and $z_i \neq 0$ for some $i \in \{1, \dots, d\}$. Then, it follows from the integral representation of $M(k_i, 2k_i + 1, \cdot)$ that

$$\begin{aligned} & \frac{1}{t^{k_i}} M(k_i, 2k_i + 1, -2\frac{z_i}{t}(z_i - \sqrt{t}y_i)) \\ &= \frac{\Gamma(2k_i + 1)}{\Gamma(k_i)\Gamma(k_i + 1)} \int_0^1 \frac{1}{t^{k_i}} e^{-2\frac{z_i}{t}(z_i - \sqrt{t}y_i)u} u^{k_i-1} (1-u)^{k_i} du \\ &= \frac{\Gamma(2k_i + 1)}{\Gamma(k_i)\Gamma(k_i + 1)} \int_0^{\frac{1}{t}} e^{-2z_i(z_i - \sqrt{t}y_i)v} v^{k_i-1} (1-tv)^{k_i} dv. \end{aligned}$$

Now, applying the Lebesgue dominated convergence theorem, we obtain

$$\lim_{t \rightarrow 0} \frac{1}{t^{k_i}} M(k_i, 2k_i + 1, -2\frac{z_i}{t}(z_i - \sqrt{t}y_i)) = \frac{\Gamma(k_i + 1)}{\sqrt{\pi} z_i^{2k_i}}.$$

Thus

$$\begin{aligned} & \lim_{t \rightarrow 0} \frac{1}{t^\gamma} e^{-\langle \frac{z}{t}, z - \sqrt{t}y \rangle} E_k \left(\frac{z}{t}, z - \sqrt{t}y \right) w_k \left(z - \sqrt{t}y \right) \\ & \geq \prod_{i=1}^d \min \left(1, y_i^{2k_i}, \frac{\Gamma(k_i + 1)}{\sqrt{\pi}} \right) =: \theta(y). \end{aligned}$$

Hence, Fatou's lemma applied to (20) yields that

$$\liminf_{t \rightarrow 0} P^z[X_t \in C \cap B(z, r)] \geq \int_{C_0} e^{-\frac{|y|^2}{2}} \theta(y) dy > 0.$$

□

4 Dirichlet Problem

This section is devoted to study the following Dirichlet problem : Giving a regular open subset D of \mathbb{R}^d and a function $f \in C(\Gamma_D)$, we shall investigate existence and uniqueness of function $h \in C(\overline{W_D}) \cap C^2(D)$ satisfying the boundary value problem

$$\begin{cases} \Delta_k h &= 0 & \text{in } D, \\ h &= f & \text{in } \Gamma_D. \end{cases} \quad (21)$$

For every square integrable functions φ and ψ on \mathbb{R}^d with respect to the measure $w_k(x)dx$, we define

$$\langle \varphi, \psi \rangle_k = \int \varphi(x) \psi(x) w_k(x) dx.$$

Lemma 8. *For every bounded open set D and for every $\varphi, \psi \in C_c^2(\mathbb{R}^d)$,*

$$\langle H_D \psi, \Delta_k \varphi \rangle_k = \langle \Delta_k \psi, H_D \varphi \rangle_k. \quad (22)$$

Proof. Applying formula (13) to ψ , we have

$$\langle H_D \psi, \Delta_k \varphi \rangle_k = -\frac{1}{2} \int G^k(z, y) \Delta_k \psi(y) w_k(y) dy H_D(x, dz) \Delta_k \varphi(x) w_k(x) dx. \quad (23)$$

Then (22) is obtained by Fubini's theorem and formulas (11) and (13). Here, since φ and ψ are with compact supports, formulas (12) and (6) justify the transformation of the integrals in (23) by Fubini's theorem. \square

A set D is called *W-invariant* if ${}^W D = D$ which, in turn, is equivalent to $\Gamma_D = \partial D$. We finally have the necessary tools at our disposal for solving the following Dirichlet problem.

Theorem 9. *Let D be a W-invariant regular open subset of \mathbb{R}^d . For every function $f \in C(\partial D)$, there exists one and only one function $h \in C(\overline{D}) \cap C^2(D)$ such that*

$$\begin{cases} \Delta_k h &= 0 & \text{in } D, \\ h &= f & \text{in } \partial D. \end{cases} \quad (24)$$

Moreover, h is given by

$$h(x) = \int_{\partial D} f(y) H_D(x, dy), \quad x \in \overline{D}.$$

Proof. In virtue of Theorem 5, we observe that for $f \in C(\partial D)$, every solution h of (21) satisfies necessarily :

$$\begin{cases} h \text{ is } X\text{-harmonic in } D, \\ h = f \text{ in } \partial D. \end{cases} \quad (25)$$

Then, by Proposition 3, (24) admits at most one solution. The function $H_D f$ is X -harmonic in D by (8). Moreover, the regularity of D yields that $H_D f$ is a continuous extension of f to \overline{D} . Therefore, according to Theorem 5, $H_D f$ will be the unique solution of (24) provided it is twice differentiable in D . On the other hand, it has been shown in [7] that Δ_k is hypoelliptic in D (see also [10]), i.e., a continuous function g in D which satisfies

$$\langle g, \Delta_k \varphi \rangle_k = 0 \quad \text{for all } \varphi \in C_c^\infty(D) \quad (26)$$

is necessary infinitely differentiable in D . Thus to complete the proof we only need to show that (26) holds true for $g = H_D f$. To this end let $\varphi \in C_c^\infty(D)$ and let $(f_n)_{n \geq 1} \subset C_c^2(\mathbb{R}^d)$ be a sequence which converges uniformly to f in ∂D . Since $H_D \varphi(y) = 0$ for all $y \in \mathbb{R}^d$, applying (22) we obtain

$$\langle H_D f_n, \Delta_k \varphi \rangle_k = 0, \quad n \geq 1. \quad (27)$$

On the other hand,

$$\sup_{x \in \overline{D}} |H_D f_n(x) - H_D f(x)| \leq \sup_{y \in \partial D} |f_n(y) - f(y)| \longrightarrow 0 \quad \text{as } n \longrightarrow \infty.$$

Hence $H_D f$ satisfies (26) by letting n tend to ∞ in (27). \square

It should be noted that the hypothesis " D is W -invariant" is only needed to get the hypoellipticity of Δ_k . For open set D which is not W -invariant, the question whether Δ_k is hypoelliptic in D or not remained open. In the case of positive answer, analogous arguments as in the proof of Theorem 9 will immediately imply that $H_D f$ is the unique solution of problem (21).

Let us notice that, using methods from harmonic analysis, M. Maslouhi and E. H. Youssfi [11] studied problem (24) in the special case where $D = B$ is the unit ball of \mathbb{R}^d . They proved that, for any $f \in C(\partial B)$, the function h given by

$$h(x) = \int_{\partial B} P_\kappa(x, y) f(y) w_k(y) \sigma(dy), \quad x \in B$$

is the unique solution of (24), where P_κ denotes the Poisson kernel introduced by C. F. Dunkl and Y. Xu [5]. Hence, our above theorem immediately yields that for every $x \in B$,

$$H_B(x, dy) = P_\kappa(x, y) w_k(y) \sigma(dy).$$

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